III. *On the Motion of a Sphere in a Viscous Liquid.*

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Communicated by Lord RAYLEIGH, *D.C.L., Sec. R.S.*

Received November 10,—Read November 24, 1887.

1. THE first problem relating to the motion of a solid body in a viscous liquid which was successfully attacked was that of a sphere, the solution of which was given by Professor STOKES in 1850, in his memoir "On the Effect of the Internal Friction of Fluids on Pendulums," 'Cambridge Phil. Soc. Trans.,' vol. 9, in the following cases: (i.) when the sphere is performing small oscillations along a straight line; (ii.) when the sphere is constrained to move with uniform velocity in a straight line; (iii.) when the sphere is surrounded by an infinite liquid and constrained to rotate with uniform angular velocity about a fixed diameter: it being supposed, in the last two cases, that sufficient time has elapsed for the motion to have become steady. In the same memoir he also discusses the motion of a cylinder and a disc. The same class of problems has also been considered by MEYER[*] and OBERBECK,[†] the latter of whom has obtained the solution in the case of the steady motion of an ellipsoid, which moves parallel to any one of its principal axes with uniform velocity. The torsional oscillations about a fixed diameter, of a sphere which is either filled with liquid or is surrounded by an infinite liquid when slipping takes place at the surface of the sphere, forms the subject of a joint memoir by HELMHOLTZ and PIOTROWSKI.[‡]

Very little appears to have been effected with regard to the solution of problems in which a viscous liquid is set in motion in any given manner and then left to itself. The solution, when the liquid is bounded by a plane which moves parallel to itself, is given by Professor STOKES at the end of his memoir referred to above; and the solutions of certain problems of two-dimensional motion have been given by STEARN.[§] In the present paper I propose to obtain the solution for a sphere moving in a viscous liquid in the following cases:—(i.) when the sphere is moving in a straight line under the action of a constant force, such as gravity; (ii.) when the sphere is surrounded by viscous liquid and is set in rotation about a fixed diameter and then left to itself.

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[*] 'Crelle, Journ. Math.,' vol. 73, p. 31.

[†] 'Crelle, Journ. Math.,' vol. 81, p. 62.

[‡] 'Wissenschaftl. Abhandl.,' vol. 1, p. 172.

[§] 'Quart. Journ. Math.,' vol. 17, p. 90.

Throughout the present investigation terms involving the squares and products of the velocity will be neglected. This is of course not strictly justifiable, unless the velocity of the sphere is slow throughout the motion. If, therefore, the velocity is not slow the results obtained can only be regarded as a first approximation; and a second approximation might be obtained by substituting the values of the component velocities hereafter obtained in the terms of the second order, and endeavouring to integrate the resulting equations. I do not, however, propose to consider this point in detail.

2. In the first place it will be convenient to show that the equations of impulsive motion of a viscous liquid are the same as those of a perfect liquid.

The general equations of motion of a viscous liquid are

$$\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} - \mathrm{X} + \frac{1}{\rho}\frac{dp}{dx} - \nu\nabla^2 u = 0,$$

with two similar equations, where ν is the kinematic coefficient of viscosity.

If we regard an impulsive force as the limit of a very large finite force which acts for a very short time τ, and if we integrate the above equation between the limits τ and 0, all the integrals will vanish except those in which the quantity to be integrated becomes infinite when τ vanishes; we thus obtain

$$u - u_0 + \frac{1}{\rho}\frac{d}{dx}\int_0^\tau p\,d\tau = 0.$$

Putting $\int_0^\tau p\,d\tau = \varpi$ where ϖ is the impulsive pressure at any point of the liquid, we obtain

$$\rho(u - u_0) + \frac{d\varpi}{dx} = 0, \text{ \&c., \&c.,}$$

which are the same equations as those which determine the impulsive pressure at any point of a perfect liquid.

3. Let us now suppose that a sphere of radius a, is surrounded by a viscous liquid which is initially at rest, and let the sphere be constrained to move with uniform velocity V, in a straight line. If the squares and products of the velocity of the liquid are neglected, Professor STOKES has shown that the current function ψ must satisfy the differential equation

$$\mathrm{D}\left(\mathrm{D} - \frac{1}{\nu}\frac{d}{dt}\right)\psi = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\mathrm{D} = \frac{d^2}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\operatorname{cosec}\theta\frac{d}{d\theta}\right)$$

and (r, θ) are polar coordinates of a point referred to the centre of the sphere as origin.

Let R, Θ be the component velocities of the liquid along and perpendicular to the radius vector; then, if we assume that no slipping takes place at the surface of the sphere, the surface conditions are

$$\mathrm{R} = \frac{1}{a^2 \sin\theta}\frac{d\psi}{d\theta} = \mathrm{V}\cos\theta, \quad \ldots \ldots \ldots \quad (2)$$

$$\Theta = -\frac{1}{a \sin\theta}\frac{d\psi}{dr} = -\mathrm{V}\sin\theta. \quad \ldots \ldots \ldots \quad (3)$$

Also, at infinity R and Θ must both vanish.

These equations can be satisfied by putting

$$\psi = (\psi_1 + \psi_2)\sin^2\theta, \quad \ldots \ldots \ldots \quad (4)$$

where ψ_1 and ψ_2 are functions of r and t, which respectively satisfy the equations

$$\frac{d^2\psi_1}{dr^2} - \frac{2\psi_1}{r^2} = 0, \quad \ldots \ldots \ldots \quad (5)$$

$$\frac{d^2\psi_2}{dr^2} - \frac{2\psi_2}{r^2} = \frac{1}{\nu}\frac{d\psi_2}{dt}. \quad \ldots \ldots \ldots \quad (6)$$

The proper solution of (5) is $\psi_1 = f(t)/r$, which it will be convenient to write in the form

$$\psi_1 = \frac{\sqrt{\pi}}{2r\sqrt{\nu t}}\int_0^\infty \chi(\alpha)\,\exp.\,(-\alpha^2/4\nu t)\,d\alpha, \quad \ldots \ldots \quad (7)$$

where $\chi(\alpha)$ is an arbitrary function, which will hereafter be determined.

In order to obtain the solution of (6), let us put $\psi_2 = r\epsilon^{-\lambda^2/\nu t}\,dw/dr$, where w is a function of r alone; substituting in (6), and integrating, we obtain

$$rw = \mathrm{A}\cos\lambda\,(r - a + \alpha),$$

where a is the radius of the sphere and A and α are the constants of integration. Whence a particular solution of (6) is

$$\psi_2 = \mathrm{A}r\frac{d}{dr}\frac{\epsilon^{-\lambda^2\nu t}}{r}\cos\lambda\,(r - a + \alpha).$$

Integrating this with respect to λ between the limits ∞ and 0, and then changing A into $\mathrm{F}(\alpha)$ and integrating the result with respect to α between the same limits, we obtain

$$\psi_2 = \frac{r\sqrt{\pi}}{2\sqrt{(\nu t)}}\frac{d}{dr}\int_0^\infty \frac{\mathrm{F}(\alpha)}{r}\,\exp.\left\{-\frac{(r - a + \alpha)^2}{4\nu t}\right\}d\alpha.$$

Performing the differentiation and then integrating by parts, we obtain

$$\psi_2 = -\tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty \left\{\frac{F(\alpha)}{r} + F'(\alpha)\right\} \exp.\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\} d\alpha$$

$$+ \tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\left[F(\alpha)\exp.\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}\right]_0^\infty .$$

We shall presently show that it is possible to determine $F(\alpha)$, so that $F(0) = 0$, and $F(\alpha)\epsilon^{-\alpha^2} = 0$ when $\alpha = \infty$; hence the term in square brackets will vanish at both limits, and we obtain

$$\psi = \frac{\sin^2\theta}{2r}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty \chi(\alpha)\exp.\left(-\frac{\alpha^2}{4\nu t}\right) d\alpha$$

$$-\frac{\sin^2\theta}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty \left\{\frac{F(\alpha)}{r} + F'(\alpha)\right\}\exp.\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\} d\alpha. \quad . \quad . \quad (8)$$

We must now determine the functions χ and F so as to satisfy the surface conditions (2) and (3).

Equation (2) will be satisfied if

$$\chi(\alpha) - F(\alpha) - aF'(\alpha) = \frac{Va^3}{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equation (3) requires that

$$Va^3 = -\tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty \chi(\alpha)\exp.\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha + \tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty F(\alpha)\exp.\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha$$

$$-\frac{a}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty \{F(\alpha) + aF'(\alpha)\}\frac{d}{d\alpha}\exp.\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha.$$

Integrating the last term by parts, the preceding equation becomes

$$Va^3 = \tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty \{-\chi(\alpha) + F(\alpha) + aF'(\alpha) + a^2F''(\alpha)\}\exp.\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha, \quad . \quad (10)$$

provided, $\{F(\alpha) + aF'(\alpha)\}\exp.(-\alpha^2/4\nu t)$ vanishes at both limits. This requires that $F(0) = F'(0) = 0$, and that $F(\alpha)\epsilon^{-\alpha^2}$ and $F'(\alpha)\epsilon^{-\alpha^2}$ should each vanish when $\alpha = \infty$. When this is the case (10) will be satisfied if

$$-\chi(\alpha) + F(\alpha) + aF'(\alpha) + a^2F''(\alpha) = \frac{2Va^3}{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Whence by (9)

$$F''(\alpha) = \frac{3Va}{\pi}$$

and, therefore,

$$F(\alpha) = \frac{3Va\alpha^2}{2\pi} + C\alpha + D.$$

The conditions that $F(0) = F'(0) = 0$ require that $C = D = 0$; whence

$$F(\alpha) = \frac{3Va\alpha^2}{2\pi}, \qquad \chi(\alpha) = \frac{Va}{\pi}\left(\frac{3\alpha^2}{2} + 3a\alpha + a^2\right).$$

Also the preceding value of $F(\alpha)$ satisfies the conditions that $F(\alpha)\epsilon^{-\alpha^2}$, and $F'(\alpha)\epsilon^{-\alpha^2}$ should each vanish when $\alpha = \infty$; whence all the conditions are satisfied, and we finally obtain

$$\psi = \frac{Va\sin^2\theta}{2r\sqrt{(\pi\nu t)}}\int_0^\infty\left(\frac{3\alpha^2}{2} + 3a\alpha + \frac{a^2}{2}\right)\exp.\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha$$
$$- \frac{3Va\sin^2\theta}{2\sqrt{(\pi\nu t)}}\int_0^\infty\left(\frac{\alpha^2}{2r} + \alpha\right)\exp.\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}d\alpha. \quad . \quad . \quad (12)$$

The first integral can be evaluated; in the second put $r - a + \alpha = 2u\sqrt{(\nu t)}$ and we obtain

$$\psi = \frac{Va\sin^2\theta}{2r}\left(3\nu t + 6a\sqrt{\frac{\nu t}{\pi}} + a^2\right)$$
$$- \frac{3Va\sin^2\theta}{\sqrt{\pi}}\int_{\frac{r-a}{2\sqrt{(\nu t)}}}^\infty\left\{\frac{1}{2r}(2u\sqrt{\nu t} - r + a)^2 + 2u\sqrt{\nu t} - r + a\right\}\epsilon^{-u^2}\,du. \quad (13)$$

4. When $t = 0$ the second integral vanishes, whence the initial value of ψ is

$$\psi = \frac{Va^3\sin^2\theta}{2r},$$

which is the known value of ψ in the case of a frictionless liquid, as ought to be the case.

When t is very large, we may put $t = \infty$ in the lower limit of the second integral, which then

$$= -\frac{3Va\sin^2\theta}{r\sqrt{\pi}}\int_0^\infty\left\{2u^2\nu t + 2au\sqrt{(\nu t)} + \tfrac{1}{2}(a^2 - r^2)\right\}\epsilon^{-u^2}\,du$$

$$= -\frac{Va\sin^2\theta}{2r}\{3\nu t + 6a\sqrt{(\nu t)} + \tfrac{3}{2}(a^2 - r^2)\},$$

whence

$$\psi = \tfrac{1}{4}Va^2\sin^2\theta\left(\frac{3r}{a} - \frac{a}{r}\right).$$

This equation gives the value of ψ after a sufficient time has elapsed for the motion to have become steady, and agrees with Professor STOKES's result.

5. Let v_t be any solution of the partial differential equation

$$\phi\left(\frac{d}{dr}\right)u = \frac{du}{dt}. \qquad \qquad (14)$$

Then, if $v_0 = 0$, $\int_0^t \mathrm{F}(t-\tau)v_\tau\, d\tau$, where $\mathrm{F}(\tau)$ is any arbitrary function which is independent of r and t, and does not become infinite between the limits, will also be a solution of (14); for, substituting in (14), the right-hand side becomes

$$\mathrm{F}(0)v_t + \int_0^t \mathrm{F}'(t-\tau)v_\tau\, d\tau = \mathrm{F}(t)v_0 + \int_0^t \mathrm{F}(t-\tau)\frac{dv_\tau}{d\tau}\, d\tau$$

$$= \phi\left(\frac{d}{dr}\right)\int_0^t \mathrm{F}(t-\tau)v_\tau\, d\tau,$$

if $v_0 = 0$.

6. The second expression on the right-hand side of (13) is the value of $\psi_2 \sin^2\theta$; and it is easily seen that this expression vanishes when $t = 0$. Hence it follows that the expression which is obtained from (13) by changing t into τ and V into $\mathrm{F}'(t-\tau)\, d\tau$, and integrating the result from t to 0, is also a solution of (1). Now, if $\mathrm{F}(0) = 0$, it will be found in substituting the above-mentioned expressions in (2) and (3) that $\mathrm{F}(t)$ is the velocity of the sphere, supposing it to have started from rest; hence this expression gives the current function due to the motion of a sphere which has started from rest, and which is moving with variable velocity $\mathrm{F}(t)$.

In order to obtain the equation of motion of the sphere, we must calculate the resistance due to the liquid; but in doing this we may begin by supposing the velocity to be uniform, and perform the above-mentioned operation at a later stage of the process.

If the impressed force is a constant force, such as gravity, which acts in the direction of motion of the sphere, and Z is the resistance due to the liquid, it can be shown, as in Professor STOKES's paper, that

$$\mathrm{Z} = 2\pi a \int_0^\pi \left(pa\cos\theta - \rho\frac{d\psi_2}{dt}\sin^2\theta\right)_a \sin\theta\, d\theta,$$

and that

$$\frac{dp}{d\theta} = \rho\sin\theta\frac{d^2\psi_1}{dt\,dr} - g\rho a\sin\theta,$$

where ρ is the density of the liquid; also, since

$$\int_0^\pi p\cos\theta\sin\theta d\theta = -\tfrac{1}{2}\int_0^\pi \sin^2\theta\frac{dp}{d\theta}d\theta,$$

we obtain

$$Z = -\pi\rho a \frac{d}{dt}\int_0^\pi \left(a\frac{d\psi_1}{dr} + 2\psi_2\right)_a \sin^3\theta \, d\theta + M'g$$

$$= -\frac{M'}{a^2}\frac{d}{dt}\left(a\frac{d\psi_1}{dr} + 2\psi_2\right)_a + M'g,$$

where M′ is the mass of the liquid displaced. Now, if V were constant, we should obtain from (13)

$$a\left(\frac{d\psi_1}{dr}\right)_a = -V\left(\tfrac{3}{2}\nu t + 3a\sqrt{\frac{\nu t}{\pi}} + \tfrac{1}{2}a^2\right),$$

and

$$(\psi_2)_a = -3Va\left(\frac{\nu t}{2a} + \sqrt{\frac{\nu t}{\pi}}\right),$$

whence

$$\left(a\frac{d\psi_1}{dr} + 2\psi_2\right)_a = -V\left(\tfrac{9}{2}\nu t + 9a\sqrt{\frac{\nu t}{\pi}} + \tfrac{1}{2}a^2\right).$$

We must now change t into τ, V into $F'(t-\tau)\,d\tau$, and integrate the result with respect to τ from t to 0, and we obtain

$$Z = \frac{M'}{a^2}\frac{d}{dt}\cdot\int_0^t F'(t-\tau)\left(\tfrac{9}{2}\nu\tau + 9a\sqrt{\frac{\nu\tau}{\pi}}\right)d\tau + \tfrac{1}{2}M'\dot{v} + M'g,$$

and the equation of motion of the sphere is

$$(M + \tfrac{1}{2}M')\dot{v} + \frac{9M'}{a^2}\frac{d}{dt}\int_0^t F'(t-\tau)\left(\tfrac{1}{2}\nu\tau + a\sqrt{\frac{\nu\tau}{\pi}}\right)d\tau = (M - M')g. \qquad (15)$$

Integrating the definite integral by parts, and remembering that $F(0) = 0$, the result is

$$\int_0^t F(t-\tau)\left(\tfrac{1}{2}\nu + \tfrac{1}{2}a\sqrt{\frac{\nu}{\pi\tau}}\right)d\tau,$$

and, differentiating with respect to t, (15) becomes

$$(M + \tfrac{1}{2}M')\dot{v} + \frac{9M'}{2a^2}\left\{v\nu + a\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{F'(t-\tau)}{\sqrt{\tau}}d\tau\right\} = (M - M')g. \quad . \quad . \quad (16)$$

Let σ be the density of the sphere, and let

$$\frac{(\sigma-\rho)g}{\sigma+\frac{1}{2}\rho} = f, \qquad \frac{9\rho}{a^2(2\sigma+\rho)} = k, \qquad \lambda = k\nu, \quad . \quad . \quad . \quad . \quad (17)$$

then (16) becomes

$$\dot{v} + \lambda v + ka\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{F'(t-\tau)}{\sqrt{\tau}}\,d\tau = f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This is the equation of motion of the sphere, from which F (t) or v must be determined.

7. Up to the present time we have supposed the motion to have commenced from rest, so that F $(0) = 0$. Let us now suppose that the sphere was initially projected with velocity V. In order to obtain the equation of motion in this case we may divide the time, t, into two intervals, h and $t - h$, where h is a very small quantity, which ultimately vanishes. During the first interval* let the sphere move from rest under the action of gravity and a very large constant force, which is equal to $(M + \frac{1}{2}M')X$, and then let the large force cease to act. This force must be such as to produce a velocity, V, at the end of the interval, h, whence we must have $V = Xh$, $v = Xt$; and, therefore, $v = Vt/h$. Changing f into $f +$ X in (18), multiplying by $\epsilon^{\lambda t}$, and integrating between the limits t and 0, we obtain

$$v\epsilon^{\lambda t} = -ka\sqrt{\frac{\nu}{\pi}}\int_0^t du\int_0^u \epsilon^{\lambda u}\,F'(u-\tau)\frac{d\tau}{\sqrt{\tau}} + \int_0^h X\epsilon^{\lambda u}du + f\int_0^t \epsilon^{\lambda u}du. \quad . \quad . \quad (19)$$

Now $F'(t)$ is composed of two parts: a large part which depends upon X, and which is equal to V/h; and another part which depends upon f, and which we shall continue to denote by $F'(t)$. Hence (19) may be written

$$v\epsilon^{\lambda t} = \frac{X}{\lambda}(\epsilon^{\lambda h} - 1) + \frac{f}{\lambda}(\epsilon^{\lambda t} - 1) - ka\sqrt{\frac{\nu}{\pi}}\int_0^t du\int_0^u F'(u-\tau)\,\epsilon^{\lambda u}\frac{d\tau}{\sqrt{\tau}}$$
$$- ka\sqrt{\frac{\nu}{\pi}}\int_0^t \epsilon^{\lambda u}\chi(u)du, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where

$$\chi(u) = \int_0^u \frac{Vd\tau}{h\sqrt{\tau}}.$$

Now χ (u) depends on X, and therefore vanishes when $u > h$. When $u < h$,

$$\chi(u) = 2Vu^{\frac{1}{2}}/h;$$

therefore

$$\int_0^t \epsilon^{\lambda u}\chi(u)\,du = \int_0^h \frac{2V}{h}u^{\frac{1}{2}}\epsilon^{\lambda u}du = 0, \text{ when } h = 0.$$

Hence, in the limit when h vanishes, (20) becomes

$$v = V\epsilon^{-\lambda t} + \frac{f}{\lambda}(1 - \epsilon^{-\lambda t}) - ka\sqrt{\frac{\nu}{\pi}}\int_0^t du\int_0^u \epsilon^{-\lambda(t-u)}F'(u-\tau)\frac{d\tau}{\sqrt{\tau}}, \quad . \quad (21)$$

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* The following procedure, suggested in a Report upon this paper, has been substituted for the remainder of this section as originally written.

and the value of the acceleration is

$$\dot{v} = -V\lambda\epsilon^{-\lambda t} + f\epsilon^{-\lambda t} - ka\sqrt{\frac{\nu}{\pi}}\frac{d}{dt}\int_0^t du\int_0^u \epsilon^{-\lambda(t-u)}F'(u-\tau)\frac{d\tau}{\sqrt{\tau}}. \qquad (22)$$

8. It seems almost hopeless to attempt to determine the complete value of F from the preceding equations, but, in the case of many liquids, ν is a small quantity, and (22) and (23) may then be solved by the method of successive approximation. For a first approximation

$$\dot{v} = F'(t) = f\epsilon^{-\lambda t},$$

whence

$$\int_0^t \frac{F'(t-\tau)\,d\tau}{\sqrt{\tau}} = f\int_0^t \frac{\epsilon^{-\lambda\tau}\,d\tau}{\sqrt{(t-\tau)}}. \quad \dots\dots\dots \quad (23)$$

The integral on the right hand side of (23) cannot be evaluated in finite terms, and we shall denote it by $\phi(t)$. Putting $\tau = ty$, we obtain

$$\phi(t) = \sqrt{t}\int_0^1 \frac{\epsilon^{-\lambda ty}\,dy}{\sqrt{(1-y)}} \quad \dots\dots\dots \quad (24)$$

$$= \sqrt{t}\int_0^1 \epsilon^{-\lambda ty}\,\Sigma_0^\infty H_n y^n\,dy,$$

where

$$H_n = \frac{1.3\ldots(2n-1)}{2^n\,n!}.$$

Now

$$\int_0^1 \epsilon^{-\lambda ty}\,dy = \frac{1-\epsilon^{-\lambda t}}{\lambda t}.$$

Therefore

$$\int_0^1 y^n\epsilon^{-\lambda ty}\,dy = (-)^n\left(\frac{d}{d.\lambda t}\right)^n\frac{1-\epsilon^{-\lambda t}}{\lambda t},$$

and therefore

$$\phi(t) = \sqrt{t}\left\{\frac{1-\epsilon^{-\lambda t}}{\lambda t} + \Sigma_1^\infty(-)^n H_n\left(\frac{d}{d.\lambda t}\right)^n\frac{1-\epsilon^{-\lambda t}}{\lambda t}\right\}. \quad \dots \quad (25)$$

When t is very large we may replace $(1-\epsilon^{-\lambda t})/\lambda t$ by $(\lambda t)^{-1}$, and we shall obtain

$$\phi(t) = \frac{1}{\lambda\sqrt{t}}\left\{1 + \Sigma_1^\infty \frac{H_n}{(\lambda t)^n}\right\}$$

$$= \frac{1}{\lambda^{\frac{1}{2}}\sqrt{(\lambda t - 1)}},$$

which shows that $\phi(t) = 0$ where $t = \infty$.

Another expression for $\phi(t)$ may be obtained in the form of a series, for

$$\phi(t) = \epsilon^{-\lambda t}\int_0^t \frac{\epsilon^{\lambda\tau}\,d\tau}{\sqrt{\tau}} = 2\sqrt{t}\left\{1 - \frac{2\lambda t}{1.3} + \frac{(2\lambda t)^2}{1.3.5} - \ldots\frac{(-)^n(2\lambda t)^n}{1.3\ldots(2n+1)} + \ldots\right\}, \quad (26)$$

by successive integration by parts. The above series is convergent for all values of t, and is zero when $t = \infty$.

For a second approximation, (22) gives

$$\dot{v} = \mathrm{F}'(t) = f\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\frac{d}{dt}\int_0^t \epsilon^{-\lambda u}\phi(t-u)\,du, \quad . \quad . \quad . \quad (27)$$

and

$$v = \mathrm{V}\epsilon^{-\lambda t} + \frac{f}{\lambda}(1-\epsilon^{-\lambda t}) - fka\sqrt{\frac{\nu}{\pi}}\int_0^t \epsilon^{-\lambda u}\phi(t-u)\,du. \quad . \quad . \quad (28)$$

Let

$$\chi(t) = \frac{d}{dt}\int_0^t \epsilon^{-\lambda u}\phi(t-u)\,du, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and (27) becomes

$$\mathrm{F}'(t) = f\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\chi(t).$$

Whence to a third approximation

$$\dot{v} = -\mathrm{V}\lambda\epsilon^{-\lambda t} + f\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\chi(t) + \frac{fk^2a^2\nu}{\pi}\frac{d}{dt}\int_0^t du\int_0^u \epsilon^{-\lambda(t-u)}\chi(u-\tau)\frac{d\tau}{\sqrt{\tau}}.$$

Let

$$\psi(t) = \int_0^t \frac{\chi(\tau)d\tau}{\sqrt{(t-\tau)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and the last equation becomes

$$\dot{v} = -\mathrm{V}\lambda\epsilon^{-\lambda t} + f\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\chi(t) + \frac{fk^2a^2\nu}{\pi}\frac{d}{dt}\int_0^t \epsilon^{-\lambda u}\psi(t-u)du, \quad . \quad . \quad . \quad . \quad (31)$$

and

$$v = \frac{f}{\lambda}(1-\epsilon^{-\lambda t}) + \mathrm{V}\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\int_0^t \epsilon^{-\lambda u}\phi(t-u)\,du + \frac{fk^2a^2\nu}{\pi}\int_0^t \epsilon^{-\lambda u}\psi(t-u)du. \quad (32)$$

We must now express all the above integrals in terms of $\phi(t)$. From (29) we obtain

$$\begin{aligned}\chi(t) &= \frac{d}{dt}\int_0^t \epsilon^{-\lambda(t-u)}\phi(u)\,du \\ &= \phi(t) - \lambda\int_0^t \epsilon^{-\lambda t+\lambda u}\phi(u)\,du \\ &= \phi(t) - \lambda\epsilon^{-\lambda t}\int_0^t du\int_0^u \epsilon^{\lambda\tau}\frac{d\tau}{\sqrt{\tau}}\end{aligned}$$

by (24). Changing the order of integration, the last integral

$$= \int_0^t d\tau \int_\tau^t \epsilon^{\lambda\tau} \frac{du}{\sqrt{\tau}} = \int_0^t \epsilon^{\lambda\tau}\left(\frac{t}{\sqrt{\tau}} - \sqrt{\tau}\right) d\tau$$

$$= \epsilon^{\lambda t}\left\{\phi(t)\left(t + \frac{1}{2\lambda}\right) - \frac{\sqrt{t}}{\lambda}\right\},$$

whence

$$\chi(t) = (\tfrac{1}{2} - \lambda t)\,\phi(t) + \sqrt{t}. \quad \ldots \ldots \ldots \quad (33)$$

Substituting this value of $\chi(t)$ in (30), we obtain

$$\psi(t) = \int_0^t (\tfrac{1}{2} - \lambda\tau)\,\phi(\tau)\frac{d\tau}{\sqrt{(t-\tau)}} + \int_0^t \sqrt{\frac{\tau}{t-\tau}}\,d\tau.$$

Now

$$\int_0^t \frac{\phi(\tau)d\tau}{\sqrt{(t-\tau)}} = \int_0^t d\tau \int_0^\tau \frac{\epsilon^{-\lambda u}\,du}{\sqrt{\{(t-\tau)(\tau-u)\}}}$$

$$= \int_0^t du \int_u^t \frac{\epsilon^{-\lambda u}\,d\tau}{\sqrt{\{(t-\tau)(\tau-u)\}}}$$

$$= \pi\int_0^t \epsilon^{-\lambda u} du = \frac{\pi}{\lambda}(1 - \epsilon^{-\lambda t}), \quad \ldots \ldots \ldots \quad (34)$$

also

$$\int_0^t \frac{\tau\phi(\tau)d\tau}{\sqrt{(t-\tau)}} = \int_0^t d\tau \int_0^\tau \frac{\tau\epsilon^{-\lambda u}\,du}{\sqrt{\{(t-\tau)(\tau-u)\}}}$$

$$= \int_0^t du \int_u^t \frac{\tau\epsilon^{-\lambda u}\,d\tau}{\sqrt{\{(t-\tau)(\tau-u)\}}}$$

$$= \frac{\pi}{2}\int_0^t (t+u)\epsilon^{-\lambda u} du$$

$$= \frac{\pi}{2\lambda}\left\{t(1 - 2\epsilon^{-\lambda t}) + \frac{1}{\lambda}(1 - \epsilon^{-\lambda t})\right\}, \quad \ldots \ldots \quad (35)$$

and

$$\int_0^t \sqrt{\frac{\tau}{t-\tau}}\,d\tau = \tfrac{1}{2}\pi t, \quad \ldots \ldots \ldots \quad (36)$$

whence

$$\psi(t) = \pi t\epsilon^{-\lambda t}. \quad \ldots \ldots \ldots \quad (37)$$

Again,

$$\int_0^t \epsilon^{-\lambda u}\psi(t-u)du = \pi\epsilon^{-\lambda t}\int_0^t (t-u)\,du$$

$$= \tfrac{1}{2}\pi t^2\epsilon^{-\lambda t}, \quad \ldots \ldots \ldots \quad (38)$$

whence (31) and (32) finally become

$$v = f\epsilon^{-\lambda t} - \mathrm{V}\lambda\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\left\{(\tfrac{1}{2} - \lambda t)\phi(t) + \sqrt{t}\right\} + fk^2a^2\nu t\epsilon^{-\lambda t}(1 - \tfrac{1}{2}\lambda t), \qquad (39)$$

$$v = \frac{f}{\lambda}(1 - \epsilon^{-\lambda t}) + \mathrm{V}\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\left\{\left(t + \frac{1}{2\lambda}\right)\phi(t) - \frac{\sqrt{t}}{\lambda}\right\} + \tfrac{1}{2}fk^2a^2\nu t^2\epsilon^{-\lambda t}. \quad . \quad (40)$$

These equations determine to a third approximation the values of the acceleration and velocity of the sphere, when it is projected vertically downwards with velocity, V, and allowed to descend under the action of gravity. If the sphere is ascending the sign of g must be reversed.

If no forces are in action we must put $f = 0$, and the preceding equations give the values of $\dot{v}$ and v to a first approximation only; but, on referring to (21) and (22), it will be seen that the values of these quantities to a third approximation may be obtained in this case from (39) and (40) by changing f into $-\mathrm{V}\lambda$ and expunging the terms $f\epsilon^{-\lambda t}$ and $f\lambda^{-1}(1 - \epsilon^{-\lambda t})$. We thus obtain, since $\lambda = k\nu$,

$$\dot{v} = -\mathrm{V}k\nu\epsilon^{-\lambda t} + \frac{\mathrm{V}ak^2\nu^{\frac{3}{2}}}{\sqrt{\pi}}\{(\tfrac{1}{2} - \lambda t)\,\phi(t) + \sqrt{t}\} - \mathrm{V}a^2k^3\nu^2t\epsilon^{-\lambda t}(1 - \tfrac{1}{2}\lambda t), \qquad (41)$$

$$v = \quad \mathrm{V}\epsilon^{-\lambda t} + \frac{\mathrm{V}ak^2\nu^{\frac{3}{2}}}{\sqrt{\pi}}\left\{\left(t + \frac{1}{2\lambda}\right)\phi(t) - \frac{\sqrt{t}}{\lambda}\right\} - \tfrac{1}{2}\mathrm{V}a^2k^3\nu^2t^2\epsilon^{-\lambda t}. \quad . \quad . \quad . \quad (42)$$

9. It appears from the preceding equations that the successive terms are multiplied by some power of k as well as of ν. If k is not a very large quantity, and the velocity of the sphere is not very great, the foregoing equations may be expected to give fairly correct results; but if k is a very large quantity, it may happen that, notwithstanding the smallness of ν, $k\nu$ may be so large that some of the terms neglected may be of equal or greater importance than those retained. Now, from (17), $k = 9\rho(2\sigma + \rho)^{-1}a^{-2}$; if, therefore, the sphere is considerably denser than the liquid, k will be small provided a be not very small; but if the sphere be considerably less dense than the liquid, k will approximate towards the limit $9a^{-2}$, and this will be very large if a be small, and $k\nu$ may therefore be large. On the other hand, it should be noticed that when $k\nu$ or λ is large the quantities $\epsilon^{-\lambda t}$ and $\phi(t)$ diminish with great rapidity, and it is therefore by no means impossible that the formulæ may give a fairly accurate representation of the motion even in this case.

All that we can therefore safely infer is this, that in the case of a sphere ascending or descending in a liquid whose kinematic coefficient of viscosity is small compared with the radius of the sphere (all quantities being of course referred to the same units), the formulæ would give approximately correct results, provided the velocity of the sphere were not too great. But, in the case of small bodies descending in a highly viscous liquid, it is possible that the motion represented by the formulæ *may be* very different from the actual motion; and if this should turn out to be the fact, the solution of (18) applicable to this case must be obtained by some different method.

Equation (39) shows that after a very long time has elapsed the acceleration vanishes, and the motion becomes ultimately steady; in other words, the acceleration due to gravity is counterbalanced by the retardation due to the viscosity of the liquid. When this state of things has been reached, the terminal velocity of the sphere is

$$v = \frac{f}{\lambda} = \frac{2a^2}{9\nu}\left(\frac{\sigma}{\rho} - 1\right)g.$$

This agrees with Professor STOKES's result, who applies it to show that the viscosity of the air is sufficient to account for the suspension of the clouds.

10. We shall now consider the motion of a sphere which is surrounded by an infinite liquid, and which is rotating about a fixed diameter.

We shall begin by supposing that the angular velocity of the sphere is uniform and equal to ω, and shall endeavour to obtain an expression for the component velocity of the liquid in a plane perpendicular to the axis of rotation, on the supposition that no slipping takes place at the surface of the sphere.

Assuming that the motion of the liquid is *stable*, it is easily seen that none of the quantities can be functions of ϕ, where r, θ, and ϕ are polar coordinates referred to the centre of the sphere as origin. If, therefore, we neglect squares and products of the velocities, the component velocity, v', of the liquid, perpendicular to any plane containing the axis of rotation, is determined by the equation

$$\frac{dv'}{dt} = \nu\left\{\frac{d^2v'}{dr^2} + \frac{2}{r}\frac{dv'}{dr} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)v' - \frac{v'}{r^2\sin^2\theta}\right\},$$

and if in this equation we put $v' = v\sin\theta$, where v is a function of r and t only, the equation for v is

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} - \frac{2v}{r^2} = \frac{1}{\nu}\frac{dv}{dt}. \qquad (43)$$

The value of the tangential stress per unit of area which opposes the motion of the sphere is

$$\mathrm{T} = -\nu\rho\left(\frac{1}{r\sin\theta}\frac{d\mathrm{R}}{d\phi} + \frac{dv'}{dr} - \frac{v'}{r}\right),$$

where R is the radial velocity; but, since R is not a function of ϕ, the value of this stress depends solely on that of v'. Now Professor STOKES has pointed out that unless the motion of the sphere is exceedingly slow, the motion of the liquid will not take place in planes perpendicular to the axis of rotation, but the velocity of every particle will have a component in the plane containing the particle and this axis. But since this component does not produce any effect on the motion of the sphere, which it is our object to determine, we may confine our attention solely to the calculation of v.

In addition to (43), v must satisfy the conditions:

(i.) At the surface of the sphere $v = a\omega$ for all values of t.

(ii.) When $t = 0$, $v = 0$ for all values of r greater than a, the radius of the sphere.

Let $v = \mathrm{R}\epsilon^{-\lambda^2\nu t}$ where R is a function of R alone; substituting in (43), we obtain

$$\frac{d^2\mathrm{R}}{dr^2} + \frac{2}{r}\frac{d\mathrm{R}}{dr} - \frac{2\mathrm{R}}{r^2} + \lambda^2\mathrm{R} = 0,$$

the solution of which is

$$\mathrm{R} = \mathrm{A}\frac{d}{dr}\left\{\frac{1}{r}\cos\lambda(r - a + \alpha)\right\},$$

whence

$$v = \mathrm{A}\frac{d}{dr}\left\{\frac{\epsilon^{-\lambda^2\nu t}}{r}\cos\lambda(r - a + \alpha)\right\}.$$

Integrating this with respect to λ between the limits ∞ and 0, and then changing A into $\mathrm{F}(\alpha)$ and integrating the result with respect to α between the same limits, we obtain

$$v = \tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\frac{d}{dr}\frac{1}{r}\int_0^\infty \mathrm{F}(\alpha)\,\exp.\left\{-\frac{(r - a + \alpha)^2}{4\nu t}\right\}d\alpha.$$

Performing the differentiation and then integrating by parts, we shall obtain

$$v = -\frac{1}{2r}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty\left\{\frac{\mathrm{F}(\alpha)}{r} + \mathrm{F}'(\alpha)\right\}\exp.\left\{-\frac{(r - a + \alpha)^2}{4\nu t}\right\}d\alpha, \quad . \quad (43\mathrm{A})$$

provided $\mathrm{F}(0) = 0$ and $\mathrm{F}(\alpha)\epsilon^{-\alpha^2} = 0$ when $\alpha = \infty$.

The surface condition (i.) will be satisfied if

$$\mathrm{F}(\alpha) + a\mathrm{F}'(\alpha) = -\frac{2a^3\omega}{\pi},$$

whence

$$\mathrm{F}(\alpha) = -\frac{2a^3\omega}{\pi}(1 - \epsilon^{-\alpha/a}),$$

the constant of integration being determined so that $\mathrm{F}(0) = 0$; this value of $\mathrm{F}(\alpha)$ also satisfies the condition that $\mathrm{F}(\alpha)\epsilon^{-\alpha^2} = 0$ when $\alpha = \infty$. We therefore obtain

$$v' = \frac{a^2\omega\sin\theta}{r\sqrt{(\pi\nu t)}}\int_0^\infty\left\{\frac{a}{r} + \left(1 - \frac{a}{r}\right)\epsilon^{-\alpha/a}\right\}\exp.\left\{-\frac{(r - a + \alpha)^2}{4\nu t}\right\}d\alpha. \quad . \quad (44)$$

Putting $r - a + \alpha = 2u\sqrt{(\nu t)}$ this becomes

$$v' = \frac{2a^2\omega \sin\theta}{r\sqrt{\pi}} \int_{\frac{r-a}{2\sqrt{(\nu t)}}}^{\infty} \left\{ \frac{a}{r} + \left(1 - \frac{a}{r}\right) \exp.\left(- \frac{2u\sqrt{(\nu t)} - r + a}{\alpha}\right) \right\} \epsilon^{-u^2}\, du. \qquad (45)$$

If $r > a$ it follows that $v' = 0$ when $t = 0$. When $r = a$ and $t = 0$ the lower limit of the definite integral (45) becomes indeterminate; but since, in this case, we are to have $v' = a\omega \sin\theta$, it follows that if we put $k = r - a$ the quantities k and t must vanish in such a manner that when $k = 0$ and $t = 0$, $k/2\sqrt{(\nu t)} = 0$.

When $t = \infty$ we obtain

$$v' = \frac{a^3\omega \sin\theta}{r}. \qquad (46)$$

This equation gives the value of v' after a sufficient time has elapsed for the motion to have become steady, and agrees with Professor STOKES's result.

11. Since the tangential stress per unit of area which opposes the motion of the sphere is

$$\mathrm{T} = - \nu\rho a \frac{d}{dr}\left(\frac{v'}{r}\right)_a,$$

the opposing couple is

$$\begin{aligned} \mathrm{G} &= - 2\pi\nu\rho a^3 \int_0^{\pi} \frac{d}{dr}\left(\frac{v'}{r}\right)_a \sin^2\theta\, d\theta, \\ &= - 2\pi\nu\rho a^3 \frac{d}{dr}\left(\frac{v}{r}\right)_a \int_0^{\pi} \sin^3\theta\, d\theta, \\ &= - \tfrac{8}{3}\pi\nu\rho a^4 \frac{d}{dr}\left(\frac{v}{r}\right)_a. \end{aligned}$$

If, therefore, the sphere be acted upon by a couple, N′, its equation of motion will be

$$\tfrac{8}{15}\sigma a^5 \dot{\omega} + \mathrm{G} = \mathrm{N}',$$

or

$$\frac{\sigma a \dot{\omega}}{5\rho} - \nu \frac{d}{dr}\left(\frac{v}{r}\right)_a = \mathrm{N}, \qquad (47)$$

where

$$\mathrm{N} = 3\rho \mathrm{N}'/8a^4.$$

When the motion of the sphere commences from rest the value of v or v' cosec θ will be obtained from (45) by changing t into τ, ω into $\mathrm{F}'(t - \tau)\, d\tau$, and integrating the result with respect to τ from t to 0, where $\mathrm{F}(t)$ is the variable angular velocity of the sphere.

Now,

$$\frac{d}{dr}\left(\frac{v}{r}\right)_a = \frac{1}{a}\frac{dv}{dr} - \frac{v}{a^2}.$$

Hence, if ω were uniform we should have

$$\left(\frac{dv}{dr}\right)_a = -2\omega + \frac{2\omega}{\sqrt{\pi}}\int_0^\infty \text{exp.}\ (-2u\sqrt{\nu t}/a - u^2)\,du - \frac{a\omega}{\sqrt{(\pi\nu t)}}.$$

Putting $u + \sqrt{(\nu t)}a = \beta$, the definite integral

$$= \epsilon^{\nu t/a^2}\int_{\sqrt{(\nu t)}/a}^\infty \epsilon^{-\beta^2}\,d\beta$$

$$= \epsilon^{\nu t/a^2}\left\{\frac{\sqrt{\pi}}{2} - \frac{\sqrt{(\nu t)}}{a} + \frac{(\nu t)^{\frac{3}{2}}}{3a^3} - \dots\right\}$$

$$= \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\nu t}}{a} + \frac{\nu t\sqrt{\pi}}{2a^2} - \dots\right),$$

if νt be small; whence

$$\left(\frac{dv}{dr}\right)_a = -\omega - \frac{2\omega}{a\sqrt{\pi}}\left(\sqrt{\nu t} - \frac{\nu t\sqrt{\pi}}{2a}\right) - \frac{a\omega}{\sqrt{\pi\nu t}}.$$

Changing t into τ, and ω into $\text{F}'(t-\tau)\,dt$, (47) becomes

$$\frac{\sigma a\dot{\omega}}{5\rho} + \frac{2\nu\dot{\omega}}{a} + \frac{2\nu}{a^2\sqrt{\pi}}\int_0^t \text{F}'(t-\tau)\left(\sqrt{(\nu\tau)} - \frac{\nu\sqrt{\pi}}{2a}\right)d\tau + \sqrt{\frac{\nu}{\pi}}\int_0^t \frac{\text{F}'(t-\tau)}{\sqrt{\tau}}\,d\tau = \text{N}. \quad (48)$$

Putting

$$\frac{10\rho}{\sigma a^2} = k, \qquad k\nu = \lambda,$$

(48) becomes

$$\dot{\omega} + \lambda\omega + \frac{k\nu^{\frac{3}{2}}}{a\sqrt{\pi}}\int_0^t \text{F}'(t-\tau)\left(\sqrt{\tau} - \frac{\tau}{2a}\sqrt{\nu\pi}\right)d\tau$$

$$+ \tfrac{1}{2}ka\sqrt{\frac{\nu}{\pi}}\int_0^t \text{F}'(t-\tau)\frac{d\tau}{\sqrt{\tau}} = \tfrac{1}{2}ka\,\text{N}. \quad (49)$$

Now we have supposed the motion to have commenced from rest under the action of the couple N′; but if the sphere had initially been set in rotation with angular velocity Ω, and then left to itself, it can be shown in the same manner as in § 7 that the equation of motion would be

$$\dot{\omega} + \lambda\omega + \frac{k\nu^{\frac{3}{2}}}{a\sqrt{\pi}}\int_0^t \text{F}'(t-\tau)\left(\sqrt{\tau} - \frac{\tau}{2a}\sqrt{\nu\pi}\right)d\tau + \tfrac{1}{2}ka\sqrt{\frac{\nu}{\pi}}\int_0^t \text{F}'(t-\tau)\frac{d\tau}{\sqrt{\tau}} = 0, \quad (50)$$

where $F(0) = \Omega$. Putting $\theta(t)$ for the last two terms, and integrating, we obtain

$$\omega = \Omega\epsilon^{-\lambda t} - \int_0^t \epsilon^{-\lambda(t-u)}\theta(u)\,du, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$\dot{\omega} = -\lambda\Omega\epsilon^{-\lambda t} - \frac{d}{dt}\int_0^t \epsilon^{-\lambda(t-u)}\theta(u)\,du. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

For a first approximation we have

$$\omega = \Omega\epsilon^{-\lambda t}, \qquad \dot{\omega} = -\lambda\Omega\epsilon^{-\lambda t} = F'(t).$$

Whence, if ϕ, χ, and ψ have the same meanings as in §8, a second approximation gives

$$\dot{\omega} = F'(t) = -k\nu\Omega\epsilon^{-\lambda t} + \frac{k^2 a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\chi(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

$$\omega = \Omega\epsilon^{-\lambda t} + \frac{k^2 a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\int_0^t \epsilon^{-\lambda(t-u)}\phi(u)\,du. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

And a third approximation gives

$$\dot{\omega} = -k\nu\Omega\epsilon^{-\lambda t} + \frac{k^2 a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\chi(t) + \frac{k^2\nu^{\frac{5}{2}}}{2a\sqrt{\pi}}\frac{d}{dt}\int_0^t du\int_0^u \epsilon^{-\lambda(t-\tau)}\sqrt{\tau}\,d\tau$$
$$- \frac{k^3 a^2\nu^2}{4\pi}\frac{d}{dt}\int_0^t \epsilon^{-\lambda(t-u)}\psi(u)\,du, \quad . \quad . \quad (55)$$

$$\omega = \Omega\epsilon^{-\lambda t} + \frac{k^2 a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\int_0^t \epsilon^{-\lambda(t-u)}\phi(u)\,du + \frac{k^2\nu^{\frac{5}{2}}}{2a\sqrt{\pi}}\int_0^t du\int_0^u \epsilon^{-\lambda(t-\tau)}\sqrt{\tau}\,d\tau$$
$$- \frac{k^3 a^2\nu^2}{4\pi}\int_0^t \epsilon^{-\lambda(t-u)}\psi(u)\,du. \quad . \quad . \quad (56)$$

Now we have shown in §8 that

$$\int_0^t \epsilon^{-\lambda(t-u)}\phi(u)\,du = \phi(t)\left(t + \frac{1}{2\lambda}\right) - \frac{\sqrt{t}}{\lambda}.$$

Also

$$\int_0^t du\int_0^u \epsilon^{-\lambda(t-\tau)}\sqrt{\tau}\,d\tau = \int_0^t d\tau\int_\tau^t \epsilon^{-\lambda(t-\tau)}\sqrt{\tau}\,du$$
$$= \epsilon^{-\lambda t}\int_0^t \epsilon^{\lambda\tau}\left(t\sqrt{\tau} - \tau^{\frac{3}{2}}\right)d\tau$$
$$= -\frac{1}{2\lambda}\left\{t\phi(t) - \frac{3\sqrt{t}}{\lambda} + \frac{3}{2\lambda}\phi(t)\right\}.$$

And the value of the last integral in (56) is given by (38); whence

$$\omega = \Omega\epsilon^{-\lambda t} + \frac{k^2 a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\left\{\phi(t)\left(t + \frac{1}{2\lambda}\right) - \frac{\sqrt{t}}{\lambda}\right\}$$
$$+ \frac{k^2\nu^{\frac{5}{2}}}{4a\sqrt{\pi}}\left\{\frac{3\sqrt{t}}{\lambda^2} - \phi(t)\left(\frac{3}{2\lambda^2} + \frac{t}{\lambda}\right)\right\} - \tfrac{1}{8}k^3a^2\nu^2t^2\epsilon^{-\lambda t}, \quad . \quad (57)$$

which determines the value of the angular velocity as far as $\nu^{\frac{5}{2}}$.

12. Helmholtz and Piotrowski discovered from their experiments that in the case of many liquids slipping takes place at the surface of the solid; when this happens, the surface condition is

$$\beta(v - a\omega) = \nu\rho a\frac{d}{dr}\left(\frac{v}{r}\right)_a \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (58)$$

where β is the coefficient of sliding friction. Putting $k = \nu\rho\beta^{-1}$, we obtain from (43A)

$$\frac{d}{dr}\left(\frac{v}{r}\right)_a = \tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty\left\{\frac{3\text{F}(\alpha)}{a^4} + \frac{2\text{F}'(\alpha)}{a^3}\right\}\text{exp.}\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha$$
$$- \frac{1}{2a^2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty\left\{\frac{\text{F}(\alpha)}{a} + \text{F}'(\alpha)\right\}\frac{d}{d\alpha}\text{exp.}\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha$$
$$= \frac{1}{2a^2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty\left(\text{F}'' + \frac{3\text{F}'}{a} + \frac{3\text{F}}{a^2}\right)\text{exp.}\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha,$$

provided $\text{F}(0) = \text{F}'(0) = 0$, and $\text{F}(\alpha)\epsilon^{-\alpha^2}$ and $\text{F}'(\alpha)\epsilon^{-\alpha^2}$ are zero when $\alpha = \infty$. Equation (58) will be satisfied if

$$\text{F}'' + \left(\frac{3}{a} + \frac{1}{k}\right)\text{F}' + \left(\frac{3}{a^2} + \frac{1}{ka}\right)\text{F} = \frac{2a^2\omega}{\pi k},$$

the solution of which is

$$\text{F} = -\frac{2a^4\omega}{(3k + a)\pi} + \text{A}\epsilon^{p\alpha} + \text{B}\epsilon^{q\alpha},$$

where p and q are the roots of the equation

$$x^2 + \left(\frac{3}{a} + \frac{1}{k}\right)x + \left(\frac{3}{a^2} + \frac{1}{ka}\right) = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (59)$$

The roots of (59) will be real if $a > k$, that is, if $a > \nu\rho/\beta$. Now, if there is no slipping, β will be infinite, and therefore, when there is comparatively little slipping, β will be large, and this relation will be satisfied unless a is small or ν is large; on the other hand, if there were no friction between the surface of the sphere and the liquid, β would be zero, but it seems improbable that any liquid exists which possesses the property of viscosity with regard to the internal motion of its particles, and which at the same time is incapable of exerting any action in the nature of friction against any surfaces with which it is in contact. If therefore β were zero, ν would probably also be zero, and the liquid would be frictionless. We shall therefore assume that the roots of (59) are real.

The constants A and B must be determined from the condition $\mathrm{F}(0) = \mathrm{F}'(0) = 0$, whence

$$\mathrm{F}(\alpha) = -\frac{2a^4\omega}{(3k+a)\pi}\left\{1 + \frac{q\epsilon^{p\alpha} - p\epsilon^{q\alpha}}{p-q}\right\}$$

$$\mathrm{F}'(\alpha) = -\frac{2a^2\omega}{\pi k(p-q)}(\epsilon^{p\alpha} - \epsilon^{q\alpha}),$$

also this value of F satisfies the conditions that $\mathrm{F}(\alpha)\,\epsilon^{-\alpha^2}$, and $\mathrm{F}'(\alpha)\,\epsilon^{-\alpha^2}$ should vanish when $\alpha = \infty$: whence the value of v' is

$$v' = \frac{a^2\omega\sin\theta}{r\sqrt{(\pi\nu t)}}\cdot\int_0^\infty\left[\frac{a^2}{r(3k+a)}\left(1 + \frac{q\epsilon^{p\alpha} - p\epsilon^{q\alpha}}{p-q}\right)\right.$$
$$\left. + \frac{\epsilon^{p\alpha} - \epsilon^{q\alpha}}{k(p-q)}\right]\exp.\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}d\alpha. \quad . \quad . \quad (60)$$

13. We shall lastly consider the motion of liquid contained within a sphere, which is rotating about a fixed diameter, when there is no slipping, and when the angular velocity is uniform.

In this case v must satisfy the differential equation (43), and also the condition (i.) of § 10; but (ii.) becomes $v = 0$ when $t = 0$ for all values of $r < a$: also we have a third condition, viz., that the velocity must be finite at the centre of the sphere.

A particular solution of (43), subject to the condition of finiteness at the origin, is

$$v = \tfrac{1}{2}\mathrm{A}\sqrt{\frac{\pi}{\nu t}}\frac{d}{dr}\frac{1}{r}\left[\exp.\left\{-\frac{(r-\alpha)^2}{4\nu t}\right\} - \exp.\left\{-\frac{(r+\alpha)^2}{4\nu t}\right\}\right],$$

whence if p and q are any quantities which are independent of r and t, a solution of (43) is

$$v = \tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\frac{d}{dr}\frac{1}{r}\int_q^p \mathrm{F}(\alpha)\left[\exp.\left\{-\frac{(r-\alpha)^2}{4\nu t}\right\} - \exp.\left\{-\frac{(r+\alpha)^2}{4\nu t}\right\}\right]d\alpha$$
$$= \frac{d}{dr}\frac{1}{r}\int_0^\infty d\lambda\int_q^p \mathrm{F}(\alpha)\,\epsilon^{-\lambda^2\nu t}\{\cos\lambda(r-\alpha) - \cos\lambda(r+\alpha)\}\,d\alpha.$$

If we put $p = a$, $q = 0$, $\mathrm{F}(\alpha) = \alpha$, the double integral when $t = 0$ is equal to r by FOURIER's theorem, for all values of r between a and 0. If we put $p = \infty$, $q = a$, the integral when $t = 0$ is zero for all values of r which do not lie between a and ∞. The solution of the problem is therefore contained in the formula

$$v = \tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\frac{d}{dr}\frac{1}{r}\left[\mathrm{A}\int_0^a \alpha + \int_a^\infty \mathrm{F}(\alpha)\right]\left[\exp.\left\{\frac{(r-\alpha)^2}{4\nu t}\right\} - \exp.\left\{-\frac{(r+\alpha)^2}{4\nu t}\right\}\right]d\alpha, \quad (61)$$

where A is a constant, which, together with the function $F(\alpha)$, must be so as to satisfy the conditions of the problem.

14. Though I am convinced that a solution of the problem exists in t definite integral, I have not succeeded in obtaining it; and therefo solution of a different character.

Let $S(r)$ denote the spherical function $d(r^{-1}\sin r)/dr$; then a solu subject to the condition of finiteness at the origin, is

$$v = \Sigma A_\lambda \epsilon^{-\lambda^2 \nu t} S(\lambda r) + \omega a, \quad . \quad . \quad . \quad . \quad . \quad .$$

when $r = a$, $v = \omega a$ for all values of t, whence

$$S(\lambda a) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

and the different values of λ are the roots of (63).

Initially $v = 0$, whence

$$\omega a = -\Sigma A_\lambda S(\lambda r).$$

Let λ and μ be different roots of (63), and let $T = S(\mu r)$, then, since S the equation

$$\frac{d^2S}{dr^2} + \frac{2}{r}\frac{dS}{dr} - \frac{2S}{r^2} + \lambda^2 S = 0,$$

we obtain,

$$(\lambda^2 - \mu^2)\int_0^a STr^2\,dr + \left[r^2T\frac{dS}{dr} - r^2S\frac{dT}{dr}\right]_0^a = 0,$$

and since by (63), S and T both vanish where $r = a$, we obtain

$$\int_0^a STr^2\,dr = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

provided λ and μ are different. To find the value of the integral whe $\mu = \lambda + d\lambda$; then from (64)

$$2\lambda\,d\lambda\int_0^a S^2r^2\,dr + a^2\left[S\frac{d^2S}{dr\,d\lambda} - \frac{dS}{dr}\frac{dS}{d\lambda}\right]_0^a d\lambda = 0,$$

or,

$$\int_0^a S^2r^2\,dr = \tfrac{1}{2}a^3S'^2(\lambda a), \quad . \quad . \quad . \quad . \quad . \quad .$$

where the accents denote differentiation with respect to λa; whence

$$-\tfrac{1}{2}A_\lambda a^3S'^2(\lambda a) = \frac{\omega a}{\lambda}\int_0^a \frac{d}{dr}\frac{\sin\lambda r}{r}\,dr,$$

$$= \frac{\omega}{\lambda}\sin\lambda a.$$

Therefore

$$A_\lambda = -\frac{2\omega(\sin\lambda a - \lambda a)}{a^3\lambda S'^2(\lambda a)},$$

and

$$v = \omega a - \frac{2\omega}{a^3}\Sigma\frac{\epsilon^{-\lambda^2\mu t}(\sin\lambda a - \lambda a)\,S(\lambda r)}{\lambda S'^2(\lambda a)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

whence the velocity of the liquid, which is equal to $v\sin\theta$, can be found.

When the angular velocity is variable, the value of the retarding couple, and the equation of motion of the sphere, can be obtained by a process analogous to that employed in § 11.

[March 10th, 1888.—Since this paper was read, a paper has been published in the 'Quarterly Journal of Mathematics,'* by Mr. WHITEHEAD, in which he attempts to develope a method of obtaining approximate solutions of problems relating to the motion of a viscous liquid, when the terms involving the squares and products of the velocities are retained; and he applies his method (see p. 90) to obtain expressions for the components *in* the plane passing through the axis of rotation, of the velocity of a viscous liquid, which surrounds a sphere which is rotating about a fixed diameter, when the motion has become steady. It will be observed, however, that the expressions for these components contain the coefficient of viscosity as a factor in the denominator, and therefore become infinite when the liquid is frictionless. It would therefore appear that the method of approximation adopted is inapplicable to the problem considered.]

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* Vol. 23, p. 78.

ERRATUM.

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Page 63, *for* equation (67) as printed *read*

$$v = \omega r - 2\omega\Sigma\,\frac{\epsilon^{-\lambda^2\nu t}\sin\lambda a\,\mathrm{S}(\lambda r)}{\lambda^2 a\,\mathrm{S}'^2(\lambda a)}$$